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Discontinuity of Best Harmonic Approximants

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Let $D \subseteq R^2$ be the open unit disk. We consider best harmonic approximation to functions continuous on $\bar{D}$. In a basic paper, Hayman *et al.* characterized best harmonic approximants which are themselves continuous on $\bar{D}$. In this paper we give sufficient conditions and many simple examples of functions continuous on $\bar{D}$ which have no best harmonic approximants which are continuous on $\bar{D}$.

1. INTRODUCTION

In this paper we consider best approximation to continuous functions by harmonic functions. Specifically, let D be a domain in R^2 and let H be the set of harmonic functions on D. If f is a bounded function on D, let

$$\|f\| = \sup\{|f(x, y)|: (x, y) \in D\}.$$

If f is a bounded function on D, let $d = \inf\{\|f - h\|: h \in H\}$.

If $h \in H$ and satisfies $\|f - h\| = d$, then h is called a best harmonic approximant to f. If in addition h is continuous on $\bar{D}$, then h is called a continuous best harmonic approximant.

The study of best harmonic approximation was initiated by Burchard [1], who obtained existence and characterization results for approximation by continuous harmonic functions. These results were expanded in [3], which is the basis for the present paper. Before continuing a general discussion of the topic, we present Theorems 1.1, 1.2, and 1.3, which are basic. The proofs are found in [3].

THEOREM 1.1. *If f is a bounded function on D, then a best harmonic approximant exists.*

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THEOREM 1.2. *Suppose D is a Jordan domain and f is continuous on $\bar{D}$. If h is a continuous best harmonic approximant to f, then h is the unique best harmonic approximant to f.*

The proof of Theorem 1.2 depends on the notion of *linked* sets.

DEFINITION. Let A be a compact subset of $\bar{D}$ and let $\delta(A)$ be the union of the domains of A^c which lie entirely in $\bar{D}$. Then $\hat{A} = A \cup \delta(A)$ is called the *hull* of A.

We note that if D is a Jordan domain, we in fact have $\delta(A) \subseteq D$.

DEFINITION. If A_1 and A_2 are disjoint compact subsets of $\bar{D}$, then A_1 and A_2 are *linked* if $\hat{A}_1 \cap \hat{A}_2 \neq \varnothing$.

Now if h is a harmonic function on D, let

$$K^o_+ = \{(x, y) \in D\colon f(x, y) - h(x, y) = \|f - h\|\}$$

and

$$K^o_- = \{(x, y) \in D\colon f(x, y) - h(x, y) = -\|f - h\|\}.$$

If h is also continuous on $\bar{D}$, let

$$K_+ = \{(x, y) \in \bar{D}\colon f(x, y) - h(x, y) = \|f - h\|\}$$

and

$$K_- = \{(x, y) \in \bar{D}\colon f(x, y) - h(x, y) = -\|f - h\|\}.$$

Obviously if D is a bounded domain and f is continuous on $\bar{D}$, then at least one of K_+ and K_- is nonempty.

The following theorem is a characterization of continuous best harmonic approximants.

THEOREM 1.3. *Suppose D is a Jordan domain. If f is continuous on $\bar{D}$ and h is a continuous best harmonic approximant, then the sets K_+ and K_- are linked.*

Conversely, suppose h is a harmonic function on D and continuous on $\bar{D}$ such that the sets K_+ and K_- are linked. Then h is the unique best harmonic approximant to f.

In view of Theorems 1.2 and 1.3, an interesting problem would be to characterize those functions f, continuous on $\bar{D}$, which have a continuous best harmonic approximant. In [3] an example is given of a continuous function on $\bar{D}$ which has no continuous best harmonic approximant. In this paper we show that this phenomenon is common and includes some very simple functions. We do this by giving a sufficient condition for real analytic functions to possess no continuous best harmonic approximant.

See also the papers [2, 7, 8] for related results.

We finish this section with three corollaries to Theorem 1.3 concerning some cases when f has a constant best harmonic approximant.

COROLLARY 1.4. *Let D be a Jordan domain and f continuous on $\bar{D}$. Define*

$$m = \inf\{f(x, y): (x, y) \in \bar{D}\}$$

and

$$M = \sup\{f(x, y): (x, y) \in \bar{D}\}.$$

Let

$$E_1 = \{(x, y) \in \bar{D}: f(x, y) = m\}$$

and

$$E_2 = \{(x, y) \in \bar{D}: f(x, y) = M\}.$$

Then f has a constant best harmonic approximant if and only if $\hat{E}_1$ and $\hat{E}_2$ are linked.

Proof. The theorem follows from the observations that any constant best harmonic approximant must be $h = \frac{1}{2}(m + M)$, and that in this case, we have $K_- = E_1$ and $K_+ = E_2$.

We next note that Corollary 1.4 can be used to characterize the *metric complement* of H, that is, the set of functions f for which 0 is the best harmonic approximant.

COROLLARY 1.5. *Using the notation of Corollary* 1.4, *f is in the metric complement of H if and only if $m = -M$ and E_1 and E_2 are linked.*

COROLLARY 1.6. *Let D be the open unit disk and suppose f is a radial function continuous on $\bar{D}$, that is, $f(x, y)$ depends only on $r = \sqrt{x^2 + y^2}$. Then f has a constant best harmonic approximant.*

Proof. Clearly the sets E_1 and E_2, as defined in Corollary 1.4, are linked, since E_1 and E_2 are circles centered at the origin.

2. FUNCTIONS WITH NO CONTINUOUS APPROXIMANTS

As mentioned in the Introduction, an example is given in [3] of a function, continuous on the closed unit disk $\bar{D}$, which has no continuous best harmonic approximant. The example is complicated. In this section we study the problem of whether f has a continuous best harmonic approximant. We show that a function as simple as $f(x, y) = x^3$ does not have a continuous best harmonic approximant on the unit disk $\bar{D}$.

In the rest of this section we assume $f(x, y)$ is a real analytic function defined on a Jordan domain D and is continuous on $\bar{D}$.

Let $F = \{\Delta f = 0\}$, $F_+ = \{\Delta f > 0\}$, $F_- = \{\Delta f < 0\}$.

F is a real analytic set and we may study F in terms of its dimension.

If $\dim F = 2$, then f is harmonic on D and hence the approximation problem is trivial.

If $\dim F = 0$, then f is either subharmonic or superharmonic on D, and the approximation problem is solved as in [3, Corollary to Theorem 3], where the best approximant is continuous.

Hence we assume $\dim F = 1$.

Now let h be a best harmonic approximant to f. Let $Z = \{(x, y) \in D: \nabla(f - h) = \bar{0}\}$, the set of critical points of $f - h$ in D.

Z could be empty, but note that we have $K^o_{\pm} \subseteq Z$.

In the remainder of this section, we look specifically at the function $f(x, y) = x^3$ defined on the unit disk. We show that f has no continuous best harmonic approximant. We will also indicate how the argument applies to similar functions. In the next section we will adapt the argument to include a wider class of functions. The Hopf lemma is important in this section. See [6, p. 240].

LEMMA 2.1 (Hopf). *Let D be an open set with boundary ∂D which is smooth at $(x_o, y_o) \in \partial D$. Let $u(x, y) \in C^2(D) \cap C^1(\bar{D})$, $\Delta u \geqslant 0$ on D, and $u(x_o, y_o) = \max\{u(x, y): (x, y) \in \bar{D}\}$. If u is not a constant, then the outer normal derivative $(\partial u/\partial v)(x_o, y_o) > 0$.*

LEMMA 2.2. *For the function $f(x, y) = x^3$ defined on the unit disk D, we have $K^o_+ \subseteq F_-$ and $K^o_- \subseteq F_+$.*

Proof. We have $\Delta f = 6x$, hence $F = \{x = 0\} \cap D$, $F_+ = \{x > 0\} \cap D$, and $F_- = \{x < 0\} \cap D$. Now suppose first that $K^o_+ \cap F_+ \neq \emptyset$. In F_+, $f - h$ is subharmonic. By the maximum principle, $f - h$ is constant on F_+, implying $\Delta f = 0$ on F_+, a contradiction.

Now suppose $K^o_+ \cap F \neq \emptyset$. Let $(0, y_o) \in K^o_+ \cap F$. Now we have $f - h$ is subharmonic in F_+ with a maximum value at $(0, y_o)$. By the Hopf lemma, we have $(\partial(f-h)/\partial v)(0, y_o)>0$. But $(0, y_o)\in Z$, which means $\nabla(f-h)(0, y_o)=0$, implying all directional derivatives of $f-h$ are 0 at $(0, y_o)$, a contradiction.

It follows that $K^o_+ \subseteq F_-$. The argument that $K^o_- \subseteq F_+$ is similar.

LEMMA 2.3. *Let D be a domain symmetric about the y-axis. Let f be a bounded function on D satisfying $f(-x, y)=-f(x, y)$. Then there exists a best harmonic approximant h on D satisfying $h(-x, y)=-h(x, y)$.*

Proof. Let g be a best harmonic approximant as guaranteed by Theorem 1.1. We have

$$-d\leqslant f(x, y)-g(x, y)\leqslant d,$$

$$-d\leqslant f(-x, y)-g(-x, y)\leqslant d,$$

$$-d\leqslant -f(x, y)-g(-x, y)\leqslant d,$$

$$-d\leqslant f(x, y)+g(-x, y)\leqslant d,$$

$$-d\leqslant f(x, y)-(-g(-x, y))\leqslant d,$$

for all $(x, y)\in D$. Hence $-g(-x, y)$ is a best harmonic approximant to f. Define $h(x, y)=\frac{1}{2}(g(x, y)+(-g(-x, y)))$. Then $h(x, y)$ is a best harmonic approximant to f and satisfies $h(-x, y)=-h(x, y)$.

We note that in the case of uniqueness, the best harmonic approximant h satisfies $h(-x, y)=h(x, y)$. In particular, we have

COROLLARY 2.4. *Let D be a Jordan domain symmetric about the y-axis. Let f be a function continuous on $\bar{D}$ satisfying $f(-x, y)=-f(x, y)$ on $\bar{D}$. If h is a continuous best harmonic approximant to f, then $h(-x, y)=-h(x, y)$ on $\bar{D}$.*

THEOREM 2.5. *The function $f(x, y)=x^3$ defined on the unit disk D has no continuous best harmonic approximant.*

Proof. Suppose h is a continuous best harmonic approximant to f. By Corollary 2.4, we have $h(-x, y)=-h(x, y)$ on $\bar{D}$.

By Theorem 1.3, the sets K_+ and K_- are linked, i.e., $\hat{K}_+\cap\hat{K}_-\neq\emptyset$. Since by Lemma 2.2 we have $K^o_+\subseteq F_-$ and $K^o_-\subseteq F_+$, the only way K_+ and K_- can be linked is if either K_+ contains the entire half-boundary $\partial D\cap\{x\geqslant 0\}$ or K_- contains the entire half-boundary $\partial D\cap\{x\leqslant 0\}$.

By the oddness conditions on f and h, we have that K_+ and K_- are reflections of each other across the y-axis. This implies that K_+ and K_- have non-empty intersection on ∂D, a contradiction.

Remarks. It is interesting to observe that the solution to the one variable version of the same problem does not carry over to the two variable case. In other words, if we consider the function $f(x)=x^3$ on the interval $[-1, 1]$, then the best harmonic approximant to f is $h(x)=mx$, where $m=0.75$. But $h(x, y)=(0.75)\,x$ is not the best harmonic approximant to $f(x, y)=x^3$ on the unit disk D. This would also be true of the domain D is changed to $[-1, 1]\times[-1, 1]$. This fact also provides an estimate for the distance from $f(x, y)=x^3$ to H, namely, $d<1-m=0.25$. Note also that if we consider the function $f(x, y)=x^n$, where n is an integer $\geqslant 2$, then we have

(a) no continuous best harmonic approximant if n is odd (the proof of Theorem 2.5 would apply)

(b) a continuous best harmonic approximant if n is even (f is subharmonic on D). See [3, Corollary to Theorem 3].

The argument used to prove Theorem 2.5 can be modified to apply to a more general setting. We need three more lemmas.

LEMMA 2.6. *Suppose $f\in C^\infty(D)$ (smooth). Suppose further that $(x_o, y_o)\in D$ is a critical point of f, but not an isolated critical point. Then $f_{xx}f_{yy}-f_{xy}^2=0$ at (x_o, y_o).*

Proof. Obviously we have $f_x(x_o, y_o)=f_y(x_o, y_o)=0$. By Morse theory [4, p. 8], since (x_o, y_o) is not isolated, it must be a degenerate critical point, i.e., the Jacobian $Jf(x_o, y_o)=0$. This means $f_{xx}(x_o, y_o)\, f_{yy}(x_o, y_o)-f_{xy}(x_o, y_o)^2=0$.

LEMMA 2.7. *Suppose $f\in C^\infty(D)$, that f has a local maximum (or minimum) at $(x_o, y_o)\in D$, that this maximum is not an isolated critical point of f, and that $\Delta(x_o, y_o)=0$. Then all first, second, and third partial derivatives of f vanish at (x_o, y_o).*

Proof. Without loss of generality, assume $(x_o, y_o)=(0, 0)$. Obviously we have $f_x=f_y=0$ at $(0, 0)$. By Lemma 2.6 we have $f_{xx}f_{yy}-f_{xy}^2=0$ at $(0, 0)$. But since $f_{xx}=-f_{yy}$ at $(0, 0)$, we must have $f_{xx}=f_{xy}=0$ at $(0, 0)$. Now suppose by way of contradiction that at least one third order partial derivative of f does not vanish at $(0, 0)$. Then we may write

$$f(x, y)=f(0, 0)+ax^3+bx^2y+cxy^2+dy^3+O(r^4)$$

as $r\to 0$, where at least one of a, b, c, and d is non-zero (where $r=\sqrt{x^2+y^2}$). On the line $y=mx$ we have

$$f(x, y) = f(0,0) + ax^3 + bmx^3 + cm^2x^3 + dm^3x^3 + O(r^4)$$
$$= f(0,0) + (a + bm + cm^2 + dm^3)\, x^3 + O(r^4)$$

as $r \to 0$. Choosing m so that $a + bm + cm^2 + dm^3 \neq 0$, we see that $(0, 0)$ is not a local maximum or minimum point. This contradiction implies all third partials of f vanish at $(0, 0)$.

LEMMA 2.8. *Let D be a circular sector shaped domain with angle α at the vertex (x_o, y_o). Let $u \in C^\infty(\bar{D})$, $\Delta u \geqslant 0$, $u(x_o, y_o) = \max\{u(x, y) : (x, y) \in \bar{D}\}$, and suppose that all first, second, and third partial derivatives are 0 at (x_o, y_o). If $\alpha > \pi/4$, then u is a constant.*

Proof. Without loss of generality, assume $(x_o, y_o) = (0, 0)$ and that one of the defining rays of D is the positive x-axis. Considering the plane to be the complex plane, make the change of variables $w = z^\beta$, where $\beta = \pi/\alpha$. Then $z = w^{1/\beta}$ (any branch). Define $f(w) = u(z) = u(w^{1/\beta})$. If we let $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$, we have $f(w) = u(\rho^{1/\beta} \cos \varphi/\beta,\ \rho^{1/\beta} \sin \varphi/\beta)$. Since u has a maximum value at $(0, 0)$, so does f. Also, if u is not a constant, then f is not a constant. Assuming u is not a constant, we have by the Hopf lemma

$$0 < \frac{\partial f}{\partial v}(0, 0) = \frac{\partial f}{\partial \rho}(0, 0).$$

But $\partial f/\partial \rho = u_x(\rho^{1/\beta} \cos \varphi/\beta,\ \rho^{1/\beta} \sin \varphi/\beta)(1/\beta)\, \rho^{(1/\beta)-1} \cos \varphi/\beta + u_y(\rho^{1/\beta} \cos \varphi/\beta,\ \rho^{1/\beta} \sin \varphi/\beta)(1/\beta)\, \rho^{(1/\beta)-1} \sin \varphi/\beta$.

Since all first, second, and third derivatives of u vanish at $(0, 0)$, we have

$$u(x, y) = u(0, 0) + O(r^4) \qquad \text{as} \quad r \to 0.$$

Hence

$$u_x(x, y) = O(r^4)$$

and

$$u_y(x, y) = O(r^3) \qquad \text{as} \quad r \to 0.$$

This means

$$\frac{\partial f}{\partial \rho} = O(\rho^{3/\beta})\, \rho^{(1/\beta)-1} + O(\rho^{3/\beta})\, \rho^{(1/\beta)-1}$$
$$= O(\rho^{(4/\beta)-1}) \qquad \text{as} \quad \rho \to 0.$$

Since $(4/\beta)-1=(4\alpha/\pi)-1>0$, this implies $(\partial f/\partial \rho)(0,0)=0$, a contradiction.

THEOREM 2.9. *Let $g(x)$ be a real analytic function (other than $g(x)=ax+b$) defined on $(-1,1)$ and continuous on $[-1,1]$ and define $f(x,y)=g(x)\,y$ on the unit disk $\bar{D}$. Then f has no continuous best harmonic approximant.*

Proof. Suppose h is a continuous best harmonic approximant. By an obvious variation of Corollary 2.4, h satisfies $h(x,-y)=-h(x,y)$. We have $\Delta f(x,y)=g''(x)\,y$. Since $g''(x)$ is real analytic and not identically zero, we have the set F is a discrete set of vertical lines together with the x-axis. The sets F_+ and F_- are the vertical half-strips in between the vertical lines of F, with the x-axis dividing a strip into two pieces, one of which is a part of F_+ and the other piece is a part of F_-. It is possible that a vertical line in F is bordered on both sides by either F_+ or F_- (i.e., $g''(x)$ does not change sign at this line). With this in mind, let us change the definition of F, F_+, and F_- so that F does not include such lines, and that either F_+ or F_- does include such lines. The argument used in Lemma 2.2 can be used to show that $K_+^o \cap F_+ = \varnothing$ and that K_+^o and K_-^o do not intersect F except possibly at the points where F intersects the x-axis. The Hopf lemma (Lemma 2.1) does not apply at these corner points. But if we assume K_+^o or K_-^o goes through a corner point, then this corner point would be a non-isolated critical point of $f-h$, and so Lemma 2.7 would apply. It then follows from Lemma 2.8 that $f-h$ would be constant on a cell of K_+^o or K_-^o, a contradiction. Hence $K_+^o \subseteq F_-$ and $K_-^o \subseteq F_+$. Since K_+ and K_- must be linked, either K_+ or K_- must contain at least an entire half-boundary $\{y \leqslant 0\} \cap \partial D$ or $\{y \geqslant 0\} \cap \partial D$. By the oddness property of $f-h$, this would imply K_+ and K_- have non-empty intersection on the boundary, which is a contradiction.

Another class of functions on the unit disk $\bar{D}$ which have no continuous best harmonic approximant is the class of homogeneous polynomials which are odd in x (or y).

THEOREM 2.10. *Let $f(x,y)$ be a homogeneous polynomial of degree $n \geqslant 3$ which also satisfies $f(-x,y)=-f(x,y)$. Then f has no continuous best harmonic approximant.*

Proof. Suppose h is a continuous best harmonic approximant. Then we have $h(-x,y)=-h(x,y)$. Now we have $\Delta f(x,y)$ is a homogeneous polynomial of degree $n-2$. It follows that F is a finite union of rays emanating from the origin, and F_+ and F_- are circular sectors. While Lemma 2.8 need not apply at $(0,0)$, it is still impossible for K_+^o or K_-^o to contain $(0,0)$ since K_+^o and K_-^o are reflections of each other across the y-axis. If either K_+^o or K_-^o contained $(0,0)$, we would have K_+^o and K_-^o intersection at $(0, 0)$, which is a contradiction. It follows as in Theorem 2.5 and Theorem 2.9 that the only way K_+ and K_- can be linked is if K_+ or K_- contains an entire half-boundary $\{x \leqslant 0\} \cap \partial D$ or $\{x \geqslant 0\} \cap \partial D$. By symmetry, this implies K_+ and K_- intersect on ∂D, a contradiction.

3. MORE GENERAL RESULTS

The techniques used to prove Theorems 2.5, 2.9, and 2.10 will apply to real analytic functions $f(x, y)$ satisfying $f(-x, y) = -f(x, y)$ and for which the sets F, F_+, and F_- satisfy what we may call an *inadmissible* configuration. In order to apply Lemma 2.8, we must have that the lines in F divide D into non-empty regions F_+ and F_- and the lines in F intersect at angles $\alpha > \pi/4$. Also, an easy condition to state can be given which forces K_+ or K_- to contain an entire half-boundary in order to be linked. It is contained in the following definition.

DEFINITION. The sets F, F_+, and F_- are in an *inadmissible* configuration if

(a) F_+ and F_- are non-empty,

(b) the lines in F intersect at angles $\alpha > \pi/4$,

(c) a cell G on F_+ or F_- satisfies the condition that $\bar{G} \cap \partial D$ is non-empty and connected.

We are now in a position to prove a fairly general result concerning discontinuous harmonic approximants.

THEOREM 3.1. *Let $f(x, y)$ be a real analytic function on the unit disk D, continuous on $\bar{D}$, satisfying $f(-x, y) = -f(x, y)$. If the sets F, F_+, and F_- are in an inadmissible configuration, then f has no continuous best harmonic approximant on D.*

Proof. Suppose h is a continuous best harmonic approximant to f. Then we have $h(-x, y) = -h(x, y)$ and the sets K_+ and K_- must be linked. As in the proof of Theorem 2.9, condition (b) in the definition of inadmissible configuration implies a line in K^o_+ or K^o_- cannot go through a corner from one cell of F_- or F_+ to another. It follows that $K^o_+ \subseteq F_-$ and $K^o_- \subseteq F_+$. Conditions (a) and (c) then imply that a line in K^o_+ cannot enclose a cell in F_+ or escape to the boundary in two disconnected pieces of the same cell of F_-. A similar statement is true of a line in K^o_-. Also by the oddness property of f, the closure of a single cell in F_+ or F_- cannot intersect the boundary of D in more than a half-boundary. It follows that in order for K_+ and K_- to be linked, K_+ or K_- must contain an entire half-boundary. By the oddness property of $f-h$, K_+ and K_- are reflections of each other across the y-axis. It follows that K_+ and K_- intersect on the boundary of F, a contradiction.

Theorem 3.1 can be used to construct many examples of real analytic functions on the unit disk which have no continuous best harmonic approximant. The following lemma is also used and may be of independent interest.

LEMMA 3.2. *If* $g(x, y)=\sum_{j,k=0}^{\infty} a_{jk} x^j y^k$ *converges on the unit disk* D, *then there exists* $f(x, y)=\sum_{j,k=0}^{\infty} b_{jk} x^j y^k$ *defined on* D *such that* $\Delta f(x, y)=g(x, y)$ *on* D.

Proof. Letting $z=x+yi$, we may write $g(x, y)$ in complex variables notation as $g(x, y)=\sum_{j,k=0}^{\infty} c_{jk} z^j \bar{z}^k$. Since $g(x, y)$ is real valued, we have $g(x, y)=\bar{g}(x, y)=\sum_{j,k=0}^{\infty} \bar{c}_{jk} \bar{z}^j z^k$ for all $z \in D$. This implies $c_{jk}=\bar{c}_{jk}$ for all $0 \leqslant j, j<\infty$.

Now define $f(x, y)=4 \sum_{j,k=0}^{\infty} (c_{jk}/(j+1)(k+1))\, z^{j+1} \bar{z}^{k+1}$, which converges on D.

Recall that $\Delta f=(1/4)(\partial^2 f/\partial \bar{z}\, \partial z)$, hence $\Delta f=g$ on D. Also $\bar{f}(x, y)=4 \sum_{j,k=0}^{\infty} (\bar{c}_{jk}/(j+1)(k+1))\, \bar{z}^{j+1} z^{k+1}=4 \sum_{j,k=0}^{\infty} (c_{kj}/(j+1)(k+1))\, z^{k+1} \bar{z}^{j+1}=f(x, y)$, so $f(x, y)$ is real valued. Finally, we may rewrite f in the form $f(x, y)=\sum_{j,k=0}^{\infty} b_{jk} x^i y^k$, and each b_{jk} is real.

Now suppose the solution set to $g(x, y)=\sum_{j,k=0}^{\infty} a_{jk} x^j y^k=0$ divides the closed unit disk $\bar{D}$ into cells which are in an inadmissible configuration. Then by Lemma 3.2 we may find $f(x, y)=\sum_{j,k=0}^{\infty} b_{jk} x^j y^k$ such that $\Delta f=g$. It is easy to verify that only odd powers of x appear in $f(x, y)$, and hence $f(-x, y)=-f(x, y)$.

It follows that f has no continuous best harmonic approximant.

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